Solution for HW7

Ex 16.5: 24) Let $\mathbf{r}(r,\theta)=(r\cos\theta, r\sin\theta, r^2)$, where $1\le r\le 2$, $0\le\theta\le 2\pi$.

$\mathbf{r}_r=(\cos\theta,\sin\theta,2r)$, $\mathbf{r}_\theta=(-r\sin\theta, r\cos\theta, 0)$

$$\therefore\ \mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix} i & j & k\\ \cos\theta & \sin\theta & 2r\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}=(-2r^2\cos\theta,-2r^2\sin\theta,r)$$

$$\Rightarrow \|\mathbf{r}_r\times\mathbf{r}_\theta\|=r\sqrt{4r^2+1}$$

$$\Rightarrow A=\int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\,dr\,d\theta=2\pi\int_1^2 r\sqrt{4r^2+1}\,dr$$

$$=\frac{\pi}{4}\int_1^2\sqrt{4r^2+1}\,d(4r^2+1)=\frac{\pi(17\sqrt{17}-5\sqrt5)}{6}\ //$$

26) Using spherical coordinate, since $\rho=2$ on the sphere,

$\mathbf{r}(\phi,\theta)=(2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$.

Note that $\begin{cases} z=-1 \Rightarrow -1=2\cos\phi\Rightarrow \phi=\frac{2\pi}{3}\\ z=\sqrt3 \Rightarrow \sqrt3=2\cos\phi\Rightarrow\phi=\frac{\pi}{6}.\end{cases}$

So $\frac{\pi}{6}\le\phi\le\frac{2\pi}{3}$, $0\le\theta\le2\pi$.

For spherical coordinate, $\|\mathbf{r}_\phi\times\mathbf{r}_\theta\|=\rho^2\sin\phi=4\sin\phi$.

$$\Rightarrow A=\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}4\sin\phi\,d\phi\,d\theta=2\pi\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}4\sin\phi\,d\phi=(4+4\sqrt3)\pi\ //$$

32) a) For any fixed $x=f(u)$, (y,z) lies on a circle with radius $g(u)$.

So $r(u,v)=(f(u), g(u)\cos v, g(u)\sin v)$, $a\le u\le b$, $0\le\theta\le 2\pi$

b) Put $g(u)=u$ & $f(u)=u^2$

We have: $r(u,v)=(u^2, u\cos v, u\sin v)$, $0\le v\le 2\pi$, $u\ge0$.

34) a) $\mathbf{r}(\theta,u)=(\cosh u\cos\theta, \cosh u\sin\theta, \sinh u)$

b) $\mathbf{r}(\theta,u)=(a\cosh u\cos\theta, b\cosh u\sin\theta, c\sinh u)$

48) The surface is given by $z=\frac23 x^{\frac32}+\frac23 y^{\frac32}=f(x,y)$

$$\text{So } A=\int_0^1\int_0^1\sqrt{f_x^2+f_y^2+1}\,dx\,dy=\int_0^1\int_0^1\sqrt{1+x+y}\,dx\,dy$$

$$= \int_0^1 \left[\frac{2}{3}(1+x+y)^{\frac{3}{2}}\right]_0^1 dy = \int_0^1 \left[\frac{2}{3}(2+y)^{\frac{3}{2}} - \frac{2}{3}(1+y)^{\frac{3}{2}}\right] dy$$

$$= \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1)_{\prime\prime}$$

56) a) Put $f(u) = u$ in 32a). Replace u by x & v by θ.
Then we get $r(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$.
b) See tutorial note 6.

Ex.16.6: 36) $\vec{n} = \frac{1}{a}(x,y,z)$. $\vec{F}\cdot\vec{n} = [\frac{1}{a}(x,y,z)]\cdot[\frac{1}{a}(x,y,z)] = 1$.

So $\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S dS = \frac{\text{Surface Area of sphere}}{8} = \frac{\pi a^2}{2}_{\prime\prime}$

42) Across the top:

$\vec{n} = \frac{1}{5}(x,y,z)$, $\vec{F}\cdot\vec{n} = \frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}$

$g(x,y,z) = x^2+y^2+z^2 = 25 \Rightarrow \nabla g = (2x, 2y, 2z)$

$\Rightarrow |\nabla g| = 2\sqrt{x^2+y^2+z^2} = 10$, $\nabla g \cdot p = (2x,2y,2z)\cdot(0,0,1) = 2z$.

So $dS = \frac{10}{2z} dA = \frac{5}{z} dA$

$$\Rightarrow \iint_{top} \vec{F}\cdot\vec{n}\, dS = \iint_R \left(\frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}\right)\left(\frac{5}{z}\right) dA = \iint_R (x^2+y^2+1)\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^4 (r^2+1) r\, dr\, d\theta = 2\pi \int_0^4 (r^2+1) r\, dr = \pi \int_0^4 (r^2+1)\, d(r^2+1)$$

$= 144\pi$.

Across the bottom:

$$\iint_{bottom} \vec{F}\cdot\vec{n}\, dS = \iint_{bottom} (xz, yz, 1)\cdot(0,0,-1)\, dS$$

$$= -\iint_{bottom} dS = -\text{Area(disk)} = -16\pi$$

So $\text{Flux} = \text{Flux}_{top} + \text{Flux}_{bottom} = 128\pi_{\prime\prime}$

46) $f(x,y,z) = 4x^2+4y^2-z^2=0 \Rightarrow \nabla f = (8x,8y,-2z)$

$\Rightarrow |\nabla f| = \sqrt{64x^2+64y^2+4z^2} = 2\sqrt{16x^2+16y^2+z^2} = 2\sqrt{5}z \ (z\geq 0)$.

$\nabla f \cdot p = (8x,8y,-2z)\cdot(0,0,1) = -2z$, $|\nabla f\cdot p| = 2z$.

$\Rightarrow dS = \sqrt{5}\,dA$

$\Rightarrow I_z = \iint_S (x^2+y^2)\delta\, dS = \delta\sqrt{5}\iint_R (x^2+y^2)\,dx\,dy = \delta\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^3\,dr\,d\theta$

$= \dfrac{3\sqrt{5}\pi\delta}{2}$ //

48) Let $z = \frac{h}{a}\sqrt{x^2+y^2}$ be the cone with base radius a and height h. By symmetry, $\bar{x} = 0 = \bar{y}$.

$f(x,y,z) = h^2(x^2+y^2)-a^2z^2 = 0 \Rightarrow \nabla f = (2h^2x, 2h^2y, -2a^2z)$

$\Rightarrow |\nabla f| = 2\sqrt{h^4x^2+h^4y^2+a^4z^2} = 2\sqrt{h^2(h^2(x^2+y^2))+a^4z} = 2\sqrt{h^2a^2z^2+a^4z^2}$

$= 2az\sqrt{h^2+a^2}$. $\nabla f\cdot p = (2h^2x, 2h^2y, -2a^2z)\cdot(0,0,1) = -2a^2z$

$\Rightarrow |\nabla f\cdot p| = 2a^2z \Rightarrow dS = \frac{\sqrt{h^2+a^2}}{a}dA$

So $M = \iint_S dS = \iint_R \frac{\sqrt{h^2+a^2}}{a}dA = \frac{\sqrt{h^2+a^2}}{a}$ (Area of disk)

$= \frac{\sqrt{h^2+a^2}}{a}\pi a^2 = \pi a\sqrt{h^2+a^2}$ //

$M_{xy} = \iint_S z\,dS = \frac{\sqrt{h^2+a^2}}{a}\iint_R \frac{h}{a}\sqrt{x^2+y^2}\,dx\,dy$

$= \frac{\sqrt{h^2+a^2}}{a}\int_0^{2\pi}\int_0^a \left(\frac{h}{a}\right) r^2\,dr\,d\theta = \frac{2\pi ah\sqrt{h^2+a^2}}{3}$

$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow$ Centroid $= \left(0,0,\frac{2h}{3}\right)$ //

Practice Problem:

Ex 16.5: 31) See tutorial notes 6.

33) a) Put $r(\theta,\phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$ into the equation of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

and show that L.H.S. = R.H.S.

Then check that $r(\theta,\phi)$ is injective.

b) $\begin{cases} r_\theta = (-a\sin\theta\cos\phi, b\cos\theta\cos\phi, 0), \\ r_\phi = (-a\cos\theta\sin\phi, -b\sin\theta\sin\phi, c\cos\phi) \end{cases}$

$$\Rightarrow |r_\theta \times r_\phi| = \begin{vmatrix} i & j & k \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$$= |(bc\cos\theta\cos^2\phi, ac\sin\theta\cos^2\phi, ab\sin\phi\cos\phi)|$$

$$= (b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi, a^2b^2\sin^2\phi\cos^2\phi)^{\frac{1}{2}}$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^{\pi} |r_\theta\times r_\phi|\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi)^{\frac{1}{2}} d\phi\, d\theta.$$

35) Let $f(x,y,z) = x^2+y^2-z^2 = 25$.

$\nabla f = (2x, 2y, -2z)$. $\nabla f(x_0, y_0, 0) = (2x_0, 2y_0, 0)$.

So the equation of the tangent plane is given by

$(2x_0, 2y_0, 0)\cdot(x-x_0, y-y_0, z-0) = 0$

$\Rightarrow \quad x_0x + y_0y = x_0^2 + y_0^2 = 25.$

Ex 16.6: 39) $g(x,y,z) = y - e^x = 0 \Rightarrow \nabla g = (-e^x, 1, 0) \Rightarrow |\nabla g| = \sqrt{1+e^{2x}}$

$\Rightarrow \vec{n} = \frac{(e^x, -1, 0)}{\sqrt{e^{2x}+1}}$ (pointing away from yz-plane)

$\vec{F}\cdot\vec{n} = (-2, 2y, z)\cdot\frac{(e^x,-1,0)}{\sqrt{e^{2x}+1}} = \frac{-2e^x - 2y}{\sqrt{e^{2x}+1}}$

$\nabla g \cdot p = (-e^x, 1, 0)\cdot(1,0,0) = -e^x \Rightarrow |\nabla g\cdot p| = e^x.$

$\Rightarrow dS = \frac{\sqrt{e^x+1}}{e^x} dA$

$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \frac{-2e^x-2y}{\sqrt{e^{2x}+1}} \left(\frac{\sqrt{e^{2x}+1}}{e^x}\right) = \iint_R -4\, dA$

$= -4$ (Area of the rectangle) $= -4$ //

47) a) Consider the upper hemisphere $f(x,y,z) = x^2+y^2+z^2 = a^2, z \geq 0$.

$\Rightarrow \nabla f = (2x, 2y, 2z) \Rightarrow |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2a$

$\nabla f \cdot p = (2x,2y,2z)\cdot(0,0,1) = 2z \quad (z \geq 0)$

$\Rightarrow dS = \frac{a}{z} dA$

$\Rightarrow I_z = \iint_{\partial R} \delta(x^2+y^2)\left(\frac{a}{z}\right) dA = a\delta \iint_R \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}} dA$

$= a\delta \int_0^{2\pi}\int_0^a \frac{r^2(r)}{\sqrt{a^2-r^2}} dr\, d\theta = 2\pi a\delta \int_0^a -r^2\, d\left(\sqrt{a^2-r^2}\right)$

$= 2\pi a\delta \left(\left[-r^2\sqrt{a^2-r^2}\right]_0^a - \int_0^a \sqrt{a^2-r^2}\, d(-r^2) \right)$

$= 2\pi a\delta \left(\left[-r^2\sqrt{a^2-r^2}\right]_0^a - \left[\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^a \right)$

$= \frac{4\pi}{3} a^4 \delta$

$\Rightarrow$ the moment of inertia of the whole sphere is $\frac{8\pi}{3} a^4 \delta$.

b) By Parallel Axis thm, $I_L = I_{c.m.} + mh^2 = I_{c.m.} + ma^2$.

$\frac{m}{2}$ = mass of half sphere $= \iint_S \delta\, dS = \delta \iint_R \frac{a}{z} dA$

$= a\delta \iint_R \frac{1}{\sqrt{a^2-x^2-y^2}} dx\, dy = a\delta \int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}} dr\, d\theta = 2\pi a^2 \delta$.

$\Rightarrow I_L = \frac{8\pi}{3} a^4\delta + 4\pi a^2 \delta a^2 = \frac{20}{3}\pi a^4 \delta$ //